Statistics Research Letters (SRL) Volume 4 2015
doi: 10.14355/srl.2015.04.002

When the Black-scholes Model Hits Zero?

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Abstract

In this letter we research on the mean extinction-time which is the expected time when the value of the asset reach zero in the classic Black-Sholes model. We resolve the stationary backward Kolmogorov equation and make a direct comparison between our predictions and the numerical simulations of stochastic differential equations.

Keywords

Black-scholes Model; Extinction-time; Stochastic Differential Equations; Monte Carlo

Introduction

The famous Black-Sholes model was the starting point of a new financial industry, and it has been a very important pillar of options trading since then. Throughout this letter S (t) denote the value of an asset at time t that may take any non-negative value. In the classic model for asset price behavior (see [3], and Chapters 6 and 7 of [7]), S (t) follows a geometric Brownian motion satisfying a linear SDE:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{1}$$

where μ and σ are two non-negative constants, μ is called the drift, and σ is the volatility that quantifies the risk. Typical values for σ lie between 0.05 and 0.5, i.e., a volatility between 5% and 50%, and the drift parameter is typically between 0.01 and 0.1.

It is possible to solve exactly (1) by means of Itô's formula with F(S, t) = ln(S) (see for instance [1, 11, 8, 7]). Its solution is

$$S(t) = S(0) \exp(\beta t + \sigma W(t)), \tag{2}$$

where β = μ – σ^2/2, and its expectation is

$$\mathrm{IE}(S(t)) = S(0)e^{Bt}. \tag{3}$$

There exist two different asymptotic behaviors:

$$\textbf{(i)}\ \text{If } \beta < 0, \text{ then Prob}\left\{\lim_{t\to\infty} |S(t)| = 0\right\} = 1.$$

$$\textbf{(ii)}\ \text{If } \beta > 0, \text{ then Prob}\left\{\lim_{t\to\infty} |S(t)| = \infty\right\} = 1.$$

Therefore, according to this model, if the volatility is sufficiently large, the asset price will eventually decay to zero with probability one, but without specifying how quickly it will do it depending on σ. In what follows, we are interested in Case (i).

Let T ≡ inf{t ≥ 0 : S (t) = 0} be the random variable that indicates the extinction-time, obviously T depends on the initial population size and we denote this dependenceby TS(0), whereas the expectation time until extinction, or mean extinction-time, is τS (0) = E(TS(0)). This mean extinction-time satisfies the following boundary value problem (BVP):

$$\begin{cases} \mu s\dfrac{d\tau}{ds} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{d^2\tau}{ds^2} = -1, \\ \tau(0) = 0, \\ \dfrac{d\tau}{ds}(M) = 0, \end{cases} \tag{4}$$

$$\alpha\frac{d^2\tau}{dz^2}+\beta\frac{d\tau}{dz}=-1, \tag{5}$$

$$\tau(z)=c_1+c_2e^{-(\beta/\alpha)z}-\frac{1}{\beta}z,$$

supposing that S (t) cannot exceed a value M. Now, we have a linear differential equation of variable coefficients; after the change of variable z = ln s we obtain

$$\alpha\frac{d^2\tau}{dz^2}+\beta\frac{d\tau}{dz}=-1, \tag{5}$$

with α = σ^2/2 > 0, whose exact solution is

$$\tau(z)=c_1+c_2e^{-(\beta/\alpha)z}-\frac{1}{\beta}z,$$

or equivalenly

$$\tau(s)=c_1+c_2s^{-\beta/\alpha}-\frac{1}{\beta}\ln s.$$

Nonetheless, it is evident that this last function cannot solve the boundary problem (4), because it is always unbounded; therefore, (4) has no solution. To avoid this difficulty, we study the time T at which S (T) = □ > 0, i.e., we will consider the mean extinction time for the new stochastic variable S˜ (t) = S (t) − □. This variable satisfies

$$d\tilde{S}(t)=\mu(\tilde{S}(t)+\epsilon)dt+\sigma(\tilde{S}(t)+\epsilon)dW(t),$$

and its mean extinction-time τ(˜s) = E(T) satisfies

$$\begin{cases} r(\tilde{s}+\epsilon)\dfrac{d\tau}{d\tilde{s}}+\dfrac{1}{2}\sigma^2(\tilde{s}+\epsilon)^2\dfrac{d^2\tau}{d\tilde{s}^2}=-1,\\ \tau(0)=0,\\ \dfrac{d\tau}{d\tilde{s}}(M)=0, \end{cases} \tag{6}$$

supposing that S˜ (t) cannot exceed a value M. We apply the change of variable z =ln(S~ + □) to this linear differential equation of variable coefficients, in order to obtain again (5). In particular, without lose of generality, we take □ = 1, for which the solution

$$\tau(\tilde{s})=\frac{\alpha}{\beta^2}(M+1)^{\beta/\alpha}\left[1-(\tilde{s}+1)^{-\beta/\alpha}\right]-\frac{1}{\beta}\ln(\tilde{s}+1), \tag{7}$$

solution that tends to τ(˜s) = ln(˜s + 1)/|β|, as M → ∞, because β < 0 and α > 0.

On the other hand, as we announced in the introduction, we have simulated (1) using the Euler-Maruyama method [9]. The algorithm may be described as follows:

Algorithm 1. *Let r and σ be the constants of the problem, △t the time-step, Tol the tolerance, and* nrun *the number of simulations. Then, we have:*

```
for each m in 1 : nrun
    Start with the initial value S_0
    for each n in 0, 1, 2 ...
        if S_n ≤ Tol
            extinction_time(m) ← n△t
            break
        end if
        Generate a random number w with normal distribution N(0, 1)
        S_{n+1} ← S_n + μS_n△t + σS_n √△t w
    end for
end for
Compute the mean and standard deviation of extinction_time
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In Figure 1, we have plotted in black the function τ(˜s) = ln(˜s + 1)/|β|, for μ = 0.01,σ = 1. The red circles located on the black curve correspond to the results from Algorithm 1, executed with time step △t = 10−3, tolerance Tol = 1, nrun = 105 and

S (0) = 10, 50, 100, 200, 300, 400. Observe that we have given Tol and □ the same values, i.e., Tol = □ = 1, to make both approaches comparable; indeed, their matching is obvious from the figure.

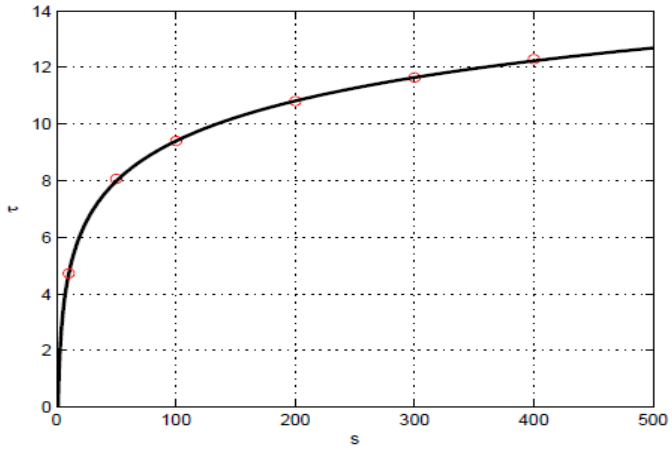

FIGURE 1 COMPARISON BETWEEN $\tau(\tilde{s}) = \ln(\tilde{s}+1)/|\beta|$ (BLACK CURVE) VERSUS THE RESULTS FROM ALGORITHM 1 (RED CIRCLES). THE MATCHING OF BOTH APPROACHES IS REMARKABLE.

Conclusions

The results obtained with both techniques are largely coincident, it is very important to underline that each execution of Algorithm 1 may take run time (and it has to be repeated again for each initial value), while function τ(˜s) is explicitly known for any point.

Moreover, when studying a real market price the volatility is not constant as assumedin the Black-Scholes model, but varies. In this way, many authors have proposed that the volatilities should be modeled by a stochastic process [6]. As already commented in [4], the mean extinction time can also be computed by solving the stationary backward Kolmogorov equations [2]. The equations are linear second-orderpartial differential equations with variable coefficients and, therefore, it is impossible in general to find exact solutions, so we can only compute numerical approximations.

On the other hand, recent results in [10] and [5] about convergence rates for the approximations

these stochastic volatility models, in my opinion, will allow to estimate and compare mean extinction-time in these more complicated stochastic models.

ACKNOWLEDGMENT

This work was supported by MEC (Spain), with the projects MTM2014-53145-P and by the Basque Government with the project ITIT641-13.

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